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AN ENGEL CONDITION WITH GENERALIZED DERIVATIONS ON MULTILINEAR POLYNOMIALS

BY

Vincenzo De Filippis*

DiSIA Facoltá di Ingegneria, Universitá di Messina
Contrada Di Dio, 98166 Messina, Italy
e-mail : defilippis@unime.it

ABSTRACT

Let R be a prime ring with extended centroid C, g a nonzero generalized derivation of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over C, I a nonzero right ideal of R.

If $[g(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)] = 0$, for all $r_1,\dots,r_n \in I$, then either $g(x) = ax$, with $(a-\gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

(i) $f(x_1,\dots,x_n)$ is central valued in $eRCe$;

(ii) $g(x) = cx + xb$, where $(c+b+\alpha)e = 0$, for $\alpha \in C$, and $f(x_1,\dots,x_n)^2$ is central valued in $eRCe$;

(iii) $char(R) = 2$ and $s_4(x_1,x_2,x_3,x_4)$ is an identity for $eRCe$.

Throughout this paper, R always denotes a prime ring with center $Z(R)$ and extended centroid C, U its right Utumi quotient ring. Here we will consider some related problems concerning generalized derivations on multilinear polynomials in prime rings. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [6], [13] for reference). By a generalized derivation on R one usually means an additive map g: $R \to R$ such that, for any $x, y \in R$, $g(xy) = g(x)y + xd(y)$, for some derivation d in R. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivation are the following: (i) $g(x) = ax + xb$, for $a, b \in R$; (ii) $g(x) = ax$, for some $a \in R$.

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The well-known theorem of Posner established that a prime ring R must be commutative if it admits a derivation d such that $[d(x), x] \in Z(R)$, for all $x \in R$, [17]. Later Lanski generalized this result to left ideals. More precisely, in [8] he proved that if R is a semiprime ring, I a nonzero left ideal, d a nonzero derivation on R and $n, t_0, t_1, \ldots, t_n$ positive integers such that the extended commutator $[d(x^{t_0}), x^{t_1}, x^{t_2}, \ldots, x^{t_n}]$ is zero for all $x \in I$, then either $d(I) = \{0\}$ or the ideal of R generated by $d(I)$ and $d(R)I$ lies in the center of R. Hence, if R is prime, then R is commutative.

More recently, in [12], Lee studied an Engel condition with derivation for polynomials on right (left) ideals of R. If you fix the attention on multilinear polynomials, then Lee's result has the following flavor: let I be a nonzero right ideal of R and $f(x_1, \ldots, x_n)$ a nonzero multilinear polynomial over C such that $[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]_k = 0$, for any $r_1, \ldots, r_n \in I$, then there exists an idempotent element $e \in soc(RC)$ such that either $f(x_1, \ldots, x_n)$ is central valued on the central simple algebra $eRCe$ or $char(R) = 2$ and $eRCe$ satisfies the standard identity s_4.

Here we will study what happens in case a similar Engel condition is satisfied by a generalized derivation g, more precisely, we will consider the case $k = 1$ and prove the following

Theorem: *Let R be a prime ring with extended centroid C, g a nonzero generalized derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, I a nonzero right ideal of R. If $[g(f(r_1, \ldots, r_n), f(r_1, \ldots, r_n)] = 0$, for all $r_1, \ldots, r_n \in I$, then either $g(x) = ax$, with $(a - \gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:*

(i) *$f(x_1, \ldots, x_n)$ is central valued in $eRCe$;*

(ii) *$g(x) = cx + xb$, where $(c + b + \alpha)e = 0$, for $\alpha \in C$, and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$;*

(iii) *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Throughout the paper, unless stated otherwise, R will be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial of R, $g \neq 0$ a generalized derivation of R and I a nonzero right ideal of R such that $[g(f(r_1, \ldots, r_n), f(r_1, \ldots, r_n)] = 0$, for all $r_1, \ldots, r_n \in I$.

For any ring S, $Z(S)$ will denote its center, and $[a, b] = ab - ba$. In addition, s_4 will denote the standard identity in 4 variables.

The related object we need to mention is the right Utumi quotient ring U of a ring R (sometimes, as in [1], U is called the maximal right ring of quotients).

The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [1]. In any case, when R is a prime ring, we need only the following properties of U:

1) $R \subseteq U$;
2) U is a prime ring;
3) The center of U, denoted by C, is a field which is called the extended centroid of R.

We also make a frequent use of the theory of generalized polynomial identities and differential identities (see [1], [3], [7], [10], [16]). In particular, we recall that when R is prime and I a nonzero right ideal of R, then I, IR and IU satisfy the same generalized polynomial identities [3].

In [13] T. K. Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $g\colon I \longrightarrow U$ such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U.

Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtain the following result.

Theorem (Theorem 3 in [13]): *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U, and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

More details about generalized derivations can be found in [6], [13], [14].

Here we begin with the following

Theorem 1: *Let R be a prime ring, $a, b \in R$ and $f(x_1, \dots, x_n)$ a noncentral multilinear polynomial over C such that*

$$[af(r_1, \dots, r_n) - f(r_1, \dots, r_n)b, f(r_1, \dots, r_n)] = 0,$$

for any $r_1, \dots, r_n \in R$. Then one of the following conclusions holds:

(i) *$a, b \in Z(R)$;*
(ii) *$f(x_1, \dots, x_n)^2$ is central valued on R and $a + b \in C$;*
(iii) *$char(R) = 2$ and R satisfies the standard identity s_4.*

Proof: Suppose that either $a \notin Z(R)$ or $b \notin Z(R)$. In this case

$$[af(x_1, \dots, x_n) - f(x_1, \dots, x_n)b, f(x_1, \dots, x_n)]$$

is a nontrivial generalized polynomial identity for R. By Theorem 2 in [2], $[af(x_1,\dots,x_n)-f(x_1,\dots,x_n)b, f(x_1,\dots,x_n)]$ is also an identity for RC. By Martindale's result in [16] RC is a primitive ring with nonzero socle. There exists a vectorial space V over a division ring D such that RC is dense of D-linear transformations over V.

If $dim_D V=\infty$, by Lemma 2 in [19], RC satisfies the following generalized identity $[ax-xb,x]$. Suppose there exists $v\in V$ such that $\{v,va\}$ are linearly D-independent. By the density of RC, there exists $w\in V$ such that $\{w,v,va\}$ are linearly D-independent and $x_0\in RC$ such that $vx_0=0$, $(va)x_0=w$ and $wx_0=v$. This leads to a contradiction, $0=v[ax_0-x_0b,x_0]=v\neq 0$. Thus for all $v\in V$, $\{v,va\}$ are linearly D-dependent, which implies that $a\in C$. From this, RC satisfies $-[xb,x]$. As above, suppose that there exists $v\in V$ such that $\{v,vb\}$ are linearly D-independent. Then there exists $y_0\in RC$ such that $vy_0=v$ and $(vb)y_0=0$. This implies that $0=-v[xb,x]=vb\neq 0$, a contradiction. Also in this case we conclude that $\{v,vb\}$ are linearly D-dependent, for all $v\in V$, and so $b\in C$.

Consider now the case $dim_D V=k$ a finite positive integer. In this case, RC is a simple ring which satisfies a nontrivial generalized polynomial identity. By [18, Theorem 2.3.29] $RC\subseteq M_t(F)$, for a suitable field F, moreover, $M_t(F)$ satisfies the same generalized identity of RC, hence $[af(r_1,\dots,r_n)-f(r_1,\dots,r_n)b, f(r_1,\dots,r_n)]=0$, for any $r_1,\dots,r_n\in M_t(F)$, moreover, $f(x_1,\dots,x_n)$ is a noncentral polynomial for $M_t(F)$. If $t=1$ there is nothing to prove. Let $t\geq 2$.

Suppose that either $char(R)\neq 2$ or R does not satisfy s_4, if not we are done.

Since $f(x_1,\dots,x_n)$ is not central then, by [15], there exist $u_1,\dots,u_n\in M_t(F)$ and $\alpha\in F-\{0\}$, such that $f(u_1,\dots,u_n)=\alpha e_{kl}$, with $k\neq l$. Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Moreover, since the set $\{f(v_1,\dots,v_n): v_1,\dots,v_n\in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i\neq j$ there exist $r_1,\dots,r_n\in M_t(F)$ such that $f(r_1,\dots,r_n)=\alpha e_{ij}$. Hence, for all $i\neq j$,

$$0=[af(r_1,\dots,r_n)-f(r_1,\dots,r_n)b, f(r_1,\dots,r_n)]=-\alpha^2 e_{ij}(a+b)e_{ij}.$$

In other words, the (j,i)-th entry of the matrix $a+b$ is zero, for all $j\neq i$. Say $a+b=c=\sum_i c_{ii}e_{ii}$, with $c_{ii}\in F$, that is c is a diagonal matrix.

Moreover, if φ is an automorphism of $M_t(F)$, the same conclusion holds for $\varphi(c)$, since as above

$$0=[\varphi(a)\varphi(f(r_1,\dots,r_n))-\varphi(f(r_1,\dots,r_n))\varphi(b), \varphi(f(r_1,\dots,r_n))]$$

Therefore, for any $i \neq j$, $\varphi(c) = (1 + e_{ij})c(1 - e_{ij})$ must be a diagonal matrix. Thus, $(c_{jj} - c_{ii})e_{ij} = 0$, that is $c_{jj} = c_{ii}$ and c is a central element. This implies that $a = -b + \gamma$, for some $\gamma \in F$. Therefore the main assumption says that

$$0 = [af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)a, f(r_1, \ldots, r_n)] = [a, f(r_1, \ldots, r_n)^2]$$

for all $r_1, \ldots, r_n \in M_t(F)$. Let G the additive subgroup generated by the polynomial $f(x_1, \ldots, x_n)^2$. By [2] $f(x_1, \ldots, x_n)^2$ is a central polynomial, unless when $[M_t(F), M_t(F)] \subseteq G$. In this last case we have that $[a, [r_1, r_2]] = 0$, for all $r_1, r_2 \in M_t(F)$. For $i \neq j$ let $[r_1, r_2] = e_{ij}$. We get $0 = ae_{ij} - e_{ij}a$ and left multiplying by e_{jj} it follows that $e_{jj}ae_{ij} = 0$, which means that the (j, i)-entry of the matrix a is zero. Therefore, a is a diagonal matrix and, as above, it is easy to prove that a is central. Then b is also central in $M_t(F)$. Therefore, in any case we get the contradiction that both a and b are central elements of R. ∎

As a natural consequence we obtain the following:

Corollary 1: *Let R be a prime ring, $a \in R$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C.*

If $[af(r_1, \ldots, r_n), f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in R$, then either $a \in Z(R)$ or $char(R) = 2$ and R satisfies the standard identity s_4.

Corollary 2: *Let R be a prime ring, $b \in R$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C.*

If $[f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in R$, then either $b \in Z(R)$ or $char(R) = 2$ and R satisfies the standard identity s_4.

Now we extend the previous results to a nonzero right ideal of R. First we recall the following notation:

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n + \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma \cdot 1)$. Thus, for a usual derivation d, we write

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n),$$

for all $r_1, \ldots, r_n \in R$.

Lemma 1: *Let R be a prime ring, g a nonzero generalized derivation of R, I a nonzero right ideal of R and $f(x_1,\dots,x_n)$ a noncentral multilinear polynomial over C such that $[g(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)] = 0$, for any $r_1,\dots,r_n \in I$. Then R satisfies a nontrivial generalized polynomial identity, unless $g(x) = ax$ and there exists $\lambda \in C$ such that $(a-\lambda)I = 0$.*

Proof: Consider the generalized derivation g assuming the form $g(x) = ax + d(x)$, for an usual derivation d of R. We divide the proof into two cases:

Case 1: Suppose that the derivation d is inner, induced by some element $q \in Q$, that is $d(x) = [q, x]$.

Thus we have, for all $r_1,\dots,r_n \in I$

$$[af(r_1,\dots,r_n) + d(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)] = [(a+q)f(r_1,\dots,r_n) - f(r_1,\dots,r_n)q, f(r_1,\dots,r_n)] = 0$$

and denote $a + q = c$, so that

$$[cf(r_1,\dots,r_n) - f(r_1,\dots,r_n)q, f(r_1,\dots,r_n)] = 0.$$

If both c and q are central elements we conclude that $g(x) = ax$, $a \in C$. Thus consider that either q or c is noncentral.

Let $u \in I$ such that $\{cu, u\}$ are linearly C-independent. If $qu = \beta u$ for some $\beta \in C$, then R satisfies

$$cf(ux_1,\dots,ux_n)^2 - \beta f(ux_1,\dots,ux_n)^2 - f(ux_1,\dots,ux_n)cf(ux_1,\dots,ux_n) + f(ux_1,\dots,ux_n)^2 q$$

which is a nontrivial GPI. On the other hand

$$[cf(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)q, f(ux_1,\dots,ux_n)]$$

is also a nontrivial GPI in case $\{q, qu\}$ are linearly C-independent.

Let now $cu = \alpha u$ for some $\alpha \in C$. Then R satisfies

$$\alpha f(ux_1,\dots,ux_n)^2 - f(ux_1,\dots,ux_n)qf(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)\alpha f(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)^2 q = -f(ux_1,\dots,ux_n)qf(ux_1,\dots,ux_n) - f(ux_1,\dots,ux_n)^2 q$$

which is again a nontrivial GPI for R.

CASE 2: Let now d be an outer derivation. Since I satisfies

$$[af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]$$

it also satisfies

$$[(a-\lambda)f(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]$$

for any $\lambda\in C$.

Note that, if there exists $\lambda\in C$ such that $(a-\lambda)I=0$, then

$$[d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]$$

is a differential identity for I. In this case, by [12], one of the following holds:

– $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$ is an identity for I, so R is a GPI-ring;
– $char(R)=2$ and $s_4(I,I,I,I)I=0$ and again R is GPI;
– $d=0$ and so $g(x)=ax$ for $(a-\lambda)I=0$, and again we are done.

Consider the case when $(a-\alpha)I\neq 0$ for all $\alpha\in C$. Since I and IU satisfy the same differential identities,

$$[af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]$$

is an identity for IU, that is, for any $u\in I$,

$$[af(ux_1,\ldots,ux_n)+d(f(ux_1,\ldots,ux_n)),f(ux_1,\ldots,ux_n)]$$

is an identity for U. In particular, pick $u\in I$ such that $au\neq\alpha u$, for all $\alpha\in C$ (it exists since $(a-\alpha)I\neq 0$). Thus U satisfies the following

$$\Big[af(ux_1,\ldots,ux_n)+f^d(ux_1,\ldots,ux_n)+\sum_i f(ux_1,\ldots,d(u)x_i+ud(x_i),\ldots,ux_n),f(ux_1,\ldots,ux_n)\Big].$$

Since d is an outer derivation, by Kharchenko's result in [7], U satisfies the identity

$$\Big[af(ux_1,\ldots,ux_n)+f^d(ux_1,\ldots,ux_n)+\sum_i f(ux_1,\ldots,d(u)x_i+uy_i,\ldots,ux_n),f(ux_1,\ldots,ux_n)\Big]$$

which is a nontrivial GPI for R, since au and u are linearly C-independent. ∎

Remark 1: Without loss of generality, R is simple and equal to its own socle, $IR = I$.

In fact, by Lemma 1, R is GPI and so RC has nonzero socle H with nonzero right ideal $J = IH$ [16]. Note that H is simple, $J = JH$, and J satisfies the same basic conditions as I. Now just replace R by H, I by J and we are done.

Remark 2: Notice that if there exists $\lambda \in C$ such that $(a-\lambda)I = 0$, then the main assumption says that

$$\begin{aligned}[](a-\lambda)f(x_1,\ldots,x_n) + d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] \\ = [d(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] \\ = 0\end{aligned}$$

for all $x_1,\ldots,x_n$ in I. In this case we obtain the required conclusions by [12].

Remark 3: It is well-known that all the following statements hold (see [11]):

(1) If $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for I, then there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $f(x_1,\ldots,x_n)$ is an identity for $eRCe$, so that a fortiori $f(x_1,\ldots,x_n)$ is central valued in $eRCe$;

(2) if $[f(x_1,\ldots,x_n), x_{n+1}]x_{n+2}$ is an identity for I then there exists $e^2 = e \in soc(RC)$ such that $IC = eRC$ and $f(x_1,\ldots,x_n)$ is central valued in $eRCe$;

(3) if $char(R) = 2$ and I satisfies $s_4(x_1,x_2,x_3,x_4)x_5$ then there exists $e^2 = e \in soc(RC)$ such that $IC = eRC$ and $s_4(x_1,\ldots,x_4)$ is an identity for $eRCe$;

(4) if $g(x) = cx + xb$ such that $(c+b+\alpha)I = 0$, for a suitable $\alpha \in C$, and I satisfies $[f(x_1,\ldots,x_n)^2, x_{n+1}]x_{n+2}$, then there exists $e^2 = e \in soc(RC)$ such that $IC = eRC$, $f(x_1,\ldots,x_n)^2$ is central valued in $eRCe$ and also $(b+c+\alpha)e = 0$.

Remark 4: Since $R = H$ is a regular ring, then for any $a_1,\ldots,a_n \in I$ there exists $h = h^2 \in R$ such that $\sum_{i=1}^{n} a_iR = hR$. Then $h \in IR = I$ and $a_i = ha_i$ for each $i = 1,\ldots,n$.

Theorem 2: *Let R be a prime ring, a, b elements of R, I a nonzero right ideal of R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C such that $[af(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)b, f(r_1,\ldots,r_n)] = 0$, for any $r_1,\ldots,r_n \in I$. If there exists a suitable $\gamma \in C$ such that $(a+b+\gamma)I = 0$, then either there exists $\lambda \in C$ such that $(a-\lambda)I = 0$ and $b \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:*

(i) *$(c+b+\gamma)e = 0$ and $f(x_1,\ldots,x_n)^2$ is central valued in $eRCe$;*

(ii) *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Proof: By $ax = -bx + \gamma x$, for all $x \in I$, we have that, for any $r_1, \ldots, r_n \in I$,

$$0 = [bf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] = [b, f(r_1, \ldots, r_n)^2].$$

By Theorem 6 in [12], it follows that either $b \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and either $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$ or $char(R) = 2$ and $s_4(eRCe) = 0$.

Moreover, if $b \in C$ we get $(a - \lambda)x = 0$ for all $x \in I$ and $\lambda = b - \gamma \in C$, in any case we are done. ∎

Continuing our line of investigation, we need the following

Lemma 2: *Let R be a prime ring, $a \in R$, I a nonzero right ideal of R and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If*

$$[af(r_1, \ldots, r_n), f(r_1, \ldots, r_n)] = 0$$

for all $r_1, \ldots, r_n \in I$, then either there exists $\gamma \in C$ such that $(a - \gamma)I = 0$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

(i) *$f(x_1, \ldots, x_n)$ is central valued in $eRCe$;*

(ii) *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Proof: Suppose by contradiction that no conclusion holds. In light of Remarks 2 and 3 there exist $b, b_1, \ldots, b_{n+2}, c_1, \ldots, c_5 \in I$ such that

- $[f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0$;
- if $char(R) = 2$, $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$;
- $\{b, ab\}$ are linearly C-independent.

By Remark 4, there exists an idempotent element $h \in IH = IR$ such that $hR = \sum_{i=1}^{n+2} b_iR + \sum_{j=1}^{5} c_jR + bR$ and $b_i = hb_i$, $c_j = hc_j$, $b = hb$ for any $i = 1, \ldots, n+2$, $j = 1, \ldots, 5$. Since $[af(hx_1, \ldots, hx_n), f(hx_1, \ldots, hx_n)]$ is satisfied by $R = H$, left multiplying by $(1 - h)$, we get that R satisfies $(1 - h)af(hx_1, \ldots, hx_n)^2$. By [4], it follows that either $(1 - h)ah = 0$ or $f(hx_1, \ldots, hx_n)h$ is a generalized identity for R. In this last case, $[f(hx_1, \ldots, hx_n), hx_{n+1}]hx_{n+2}$ is an identity for R and this contradicts with

$$[f(hb_1, \ldots, hb_n), hb_{n+1}]hb_{n+2} = [f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0.$$

Thus $(1 - h)ah = 0$, that is $ah = hah$. Therefore $[af(x_1, \ldots, x_n), f(x_1, \ldots, x_n)]$ is satisfied by hRh.

By Corollary 1, again since

$$[f(hb_1,\ldots,hb_n),hb_{n+1}]hb_{n+2} = [f(b_1,\ldots,b_n),b_{n+1}]b_{n+2} \neq 0,$$

we get either $ah \in Ch$ or $char(R) = 2$ and hRh satisfies s_4.

In the first case, if $ah \in Ch$, then there exists $\lambda \in C$ such that $ahb = (\lambda)hb$, that is $ab = \lambda b$, a contradiction.

In the second case, $s_4(hRh,hRh,hRh,hRh) = 0$ implies that

$$s_4(hR,hR,hR,hR)hR = 0,$$

and again we get a contradiction since

$$s_4(hc_1,hc_2,hc_3,hc_4)hc_5 = s_4(c_1,c_2,c_3,c_4)c_5 \neq 0. \qquad \blacksquare$$

Remark 5: Suppose that there exist $b_1,\ldots,b_{n+2} \in I$ such that

$$[f(b_1,\ldots,b_n),b_{n+1}]b_{n+2} \neq 0.$$

This obviously implies that $f(x_1,\ldots,x_n)x_{n+1}$ cannot be an identity for I and we may consider, without loss of generality $f(b_1,\ldots,b_n)b_{n+1} \neq 0$.

If you write $f(x_1,\ldots,x_n) = \sum_i t_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i$, where any t_i is a multilinear polynomial in $n-1$ variables, in which x_i never occurs, it follows that there exist $i \in \{1,2,\ldots,n\}$ such that $t_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i$ is not an identity for I and again we may choose, for example, $t_n(b_1,\ldots,b_{n-1})b_n \neq 0$.

LEMMA 3: *Let R be a prime ring, $b \in R$, I a nonzero right ideal of R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C. If*

$$[f(r_1,\ldots,r_n)b, f(r_1,\ldots,r_n)] = 0$$

for all $r_1,\ldots,r_n \in I$, then either $b \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

(i) *$f(x_1,\ldots,x_n)$ is central valued in $eRCe$;*

(ii) *$char(R) = 2$ and $s_4(x_1,x_2,x_3,x_4)$ is an identity for $eRCe$.*

Proof: The proof is similar to that of Lemma 2, but, for the sake of completeness, we prefer to explain the argument again.

Suppose by contradiction that there exist $b_1,\ldots,b_{n+2},c_1,\ldots,c_5 \in I$ such that

– $[f(b_1,\ldots,b_n),b_{n+1}]b_{n+2} \neq 0$, in particular, in light of Remark 5, then $t_n(b_1,\ldots,b_{n-1})b_n \neq 0$;

– if $char(R) = 2$, $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$.

By Remark 4, there exists an idempotent element $h \in IH = IR$ such that $hR = \sum_{i=1}^{n+2} b_i R + \sum_{j=1}^{5} c_j R$ and $b_i = hb_i$, $c_j = hc_j$, for any $i = 1, \ldots, n+2$, $j = 1, \ldots, 5$. Since $[f(hx_1, \ldots, hx_n(1-h))b, f(hx_1, \ldots, hx_n(1-h))]$ is satisfied by $R = H$, then R satisfies

$$t_n(hx_1, \ldots, hx_{n-1})hx_n(1-h)bt_n(hx_1, \ldots, hx_{n-1})hx_n(1-h)$$

and a fortiori R satisfies

$$((1-h)bt_n(hx_1, \ldots, hx_{n-1})hx_n)^3.$$

By a result in [5] we have that R satisfies $(1-h)bt_n(hx_1, \ldots, hx_{n-1})hx_n$ and by [4] it follows that $(1-h)bh = 0$, since $t_n(hb_1, \ldots, hb_{n-1})hb_n \neq 0$. Therefore, $[f(x_1, \ldots, x_n)b, f(x_1, \ldots, x_n)]$ is satisfied by hRh.

By Corollary 2, again since

$$[f(hb_1, \ldots, hb_n), hb_{n+1}]hb_{n+2} = [f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0,$$

we get either that $bh \in Ch$ or that $char(R) = 2$ and hRh satisfies s_4.

Since the last case contradicts $s_4(hc_1, hc_2, hc_3, hc_4)hc_5 \neq 0$, we have $bh \in Ch$, then there exists $\lambda \in C$ such that $bh = \lambda h$. Thus

$$0 = [f(hx_1, \ldots, hx_n)b, f(hx_1, \ldots, hx_n)] = f(hx_1, \ldots, hx_n)^2(b - \lambda).$$

In this case, because of the fact that $f(hb_1, \ldots, hb_n)hb_{n+1} \neq 0$, again by [4], we conclude that $b = \lambda \in C$. ∎

Theorem 3: *Let R be a prime ring, $a, b \in R$, I a nonzero right ideal of R and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C.*

If $[af(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)] = 0$, for any $r_1, \ldots, r_n \in I$, then either there exists $\gamma \in C$ such that $(a-\gamma)I = 0$ and $b \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

(i) *$f(x_1, \ldots, x_n)$ is central valued in $eRCe$;*
(ii) *$(a+b+\alpha)e = 0$, for $\alpha \in C$, and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$;*
(iii) *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Proof: Due to Theorem 2, if there exists $\alpha \in C$ such that $(a + b + \alpha)I = 0$ then the present theorem holds. Moreover, if there exists $\gamma \in C$ such that $(a-\gamma)I = 0$, it follows that $[f(r_1,\dots,r_n)b, f(r_1,\dots,r_n)] = 0$, for any $r_1,\dots,r_n \in I$, and, by Lemma 3, we are done.

In light of this, suppose by contradiction that there exist $v, w \in I$ such that

$$\{v, av\} \quad \text{are linearly C-independent}$$

and

$$\{w, (a+b)w\} \quad \text{are linearly C-independent.}$$

Moreover, suppose that there exist $b_1,\dots,b_{n+2}, s_1,\dots,s_{n+2}, c_1,\dots,c_5 \in I$ such that

– $[f(b_1,\dots,b_n), b_{n+1}]b_{n+2} \neq 0$;
– $[f(s_1,\dots,s_n)^2, s_{n+1}]s_{n+2} \neq 0$;
– if $char(R) = 2$, $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$.

Again there exists an idempotent element $h \in IR$ such that

$$hR = \sum_{i=1}^{n+2} b_i R + \sum_{j=1}^{n+2} s_j R + \sum_{k=1}^{5} c_k R + vR + wR$$

and $b_i = hb_i$, $s_j = hs_j$, $c_k = hc_k$, for any $i, j = 1,\dots,n+2$, $k = 1,\dots,5$, and $v = hv$, $w = hw$. Since $[af(hx_1,\dots,hx_n) - f(hx_1,\dots,hx_n)b, f(hx_1,\dots,hx_n)]$ is satisfied by R, left multiplying by $(1-h)$, we get that R satisfies

$$(1-h)af(hx_1,\dots,hx_n)^2.$$

By [4] it follows that either $(1-h)ah = 0$ or $f(hx_1,\dots,hx_n)h$ is a generalized identity for RC. Note that this last conclusion cannot occurs, because $f(hb_1,\dots,hb_n)hb_{n+1} = f(b_1,\dots,b_n)b_{n+1} \neq 0$. Thus $(1-h)ah = 0$.

Moreover, since

$$[af(hx_1,\dots,hx_n(1-h)) - f(hx_1,\dots,hx_n(1-h))b, f(hx_1,\dots,hx_n(1-h))] = 0,$$

we have that

$$[at_n(hx_1,\dots,hx_{n-1})hx_n(1-n) \\ -t_n(hx_1,\dots,hx_{n-1})hx_n(1-h)b, t(hx_1,\dots,hx_{n-1})hx_n(1-h)] = 0;$$

that is

$$-t_n(hx_1,\dots,hx_{n-1})hx_n(1-h)bt_n(hx_1,\dots,hx_{n-1})hx_n(1-h) = 0$$

and, in particular, $((1-h)bt_n(hx_1,\dots,hx_{n-1})hx_n)^3 = 0$.

By a result in [5], $(1-h)bt_n(hx_1,\dots,hx_{n-1})hx_n = 0$.

Again by [4], since $t_n(hb_1,\dots,hb_{n-1})hb_n = t_n(b_1,\dots,b_{n-1})b_n \neq 0$ (see Remark 5), we get $(1-h)bh = 0$. Therefore, $ah = hah$ and $bh = hbh$. Hence hRh is a finite dimensional simple central algebra which satisfies

$$[af(x_1,\dots,x_n) - f(x_1,\dots,x_n)b, f(x_1,\dots,x_n)].$$

By Theorem 1 it follows that one of the following holds:

(i) there exists $\gamma \in C$ such that $(a+b-\gamma)h = 0$, which contradicts with the fact that $\{w,(a+b)(hw)\} = \{w,(a+b)w\}$ are linearly C-independent.

(ii) $f(x_1,\dots,x_n)$ is central valued in hRh, then $[f(x_1,\dots,x_n),x_{n+1}]x_{n+2} = 0$ in hR and this contradicts

$$[f(hb_1,\dots,hb_n),hb_{n+1}]hb_{n+2} = [f(b_1,\dots,b_n),b_{n+1}]b_{n+2} \neq 0;$$

(iii) $ah, bh \in Ch$, that is, in particular, there exists $\alpha \in C$ such that $(a-\alpha)h = 0$. This is also a contradiction since $\{v,a(hv)\} = \{v,av\}$ are linearly C-independent. ∎

Finally we study the more general case and we need the following remark:

LEMMA 4: *Let R be a prime ring and $f(x_1,\dots,x_n)$ a multilinear polynomial over C. If, for $i = 1,\dots,n$,*

$$[f(r_1,\dots,z_i,\dots,r_n), f(r_1,\dots,r_n)] = 0$$

for all $z_i, r_1,\dots,r_n \in R$, then the polynomial $f(x_1,\dots,x_n)$ is central-valued on R except when $char(R) = 2$ and R satisfies s_4.

Proof: Let $s \in R$, then by assumption

$$[s, f(r_1,\dots,r_n)]_2 = \left[\sum_i f(r_1,\dots,[s,r_i],\dots,r_n), f(r_1,\dots,r_n)\right] = 0.$$

Hence the result follows by [9, Theorem]. ∎

THEOREM 4: *Let R be a prime K-algebra, with extended centroid C, g a nonzero generalized derivation of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over C and I a nonzero right ideal of R. If $[g(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)] = 0$, for all $r_1,\dots,r_n \in I$, then either $g(x) = ax$, with $(a-\gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:*

(i) *$f(x_1,\ldots,x_n)$ is central valued in $eRCe$;*

(ii) *$g(x)=cx+xb$, where $(c+b+\alpha)e=0$, for $\alpha\in C$, and $f(x_1,\ldots,x_n)^2$ is central valued in $eRCe$;*

(iii) *$char(R)=2$ and $s_4(x_1,x_2,x_3,x_4)$ is an identity for $eRCe$.*

Proof: As we have already remarked, every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x)=ax+d(x)$, for some $a\in U$ and a derivation d on U.

If $d=0$ we are done by Lemma 2. Thus we suppose that $d\neq 0$.

For $u\in I$, U satisfies the following differential identity

$$[af(ux_1,\ldots,ux_n)+d(f(ux_1,\ldots,ux_n)),f(ux_1,\ldots,ux_n)].$$

In light of Kharchenko's theory ([7], [10]), we divide the proof into two cases:

Case 1: Let d be the inner derivation induced by the element $q\in U$, that is $d(x)=[q,x]$, for all $x\in U$. Thus I satisfies the generalized polynomial identity

$$[af(x_1,\ldots,x_n)+qf(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)q,f(x_1,\ldots,x_n)]$$
$$=[(a+q)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)q,f(x_1,\ldots,x_n)].$$

If we denote $-q=b$ and $a+q=c$, the generalized derivation g is defined as $g(x)=cx+xb$, and we get the conclusion thanks to Theorem 3.

Case 2: Let d be an outer derivation of U and suppose that

$$[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$$

is not an identity for I and, in case $char(R)=2$, I does not satisfy $s_4(x_1,\ldots,x_4)x_5$, otherwise we are done (see Remark 3). Thus, there exist $b_1,\ldots,b_{n+2},c_1,\ldots,c_5\in I$ such that

$$[f(b_1,\ldots,b_n),b_{n+1}]b_{n+2}\neq 0,\quad s_4(c_1,\ldots,c_4)c_5\neq 0$$

and there exists $h^2=h\in soc(RC)$ such that $\sum_{i=1}^{n+2}b_iR+\sum_{j=1}^{5}c_jR=hR$, with $b_i=hb_i$, $c_j=hc_j$ for all $i=1,\ldots,n+2$, $j=1,\ldots,5$.

Since I and IU satisfy the same differential identities,

$$[af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]$$

is an identity for IU, that is,

$$[af(hx_1,\ldots,hx_n)+d(f(hx_1,\ldots,hx_n)),f(hx_1,\ldots,hx_n)]$$

is an identity for U. Thus U satisfies the following

$$\Big[af(hx_1,\dots,hx_n)+f^d(hx_1,\dots,hx_n) + \sum_i f(hx_1,\dots,d(h)x_i+hd(x_i),\dots,hx_n), f(hx_1,\dots,hx_n)\Big].$$

Since d is an outer derivation, by Kharchenko's result in [7], R satisfies the identity

$$\Big[af(hx_1,\dots,hx_n)+f^d(hx_1,\dots,hx_n) + \sum_i f(hx_1,\dots,d(h)x_i+hy_i,\dots,hx_n), f(hx_1,\dots,hx_n)\Big].$$

In particular, U satisfies the blended component

$$\Big[\sum_i f(hx_1,\dots,hy_i,\dots,hx_n), f(hx_1,\dots,hx_n)\Big]$$

so that hUh satisfies

$$[f(x_1,\dots,y_i,\dots,x_n), f(x_1,\dots,x_n)],$$

for all $i=1,\dots,n$. By Lemma 4 we have that either $f(x_1,\dots,x_n)$ is central valued in hUh or $char(R)=2$ and $s_4(hUh)=0$. In both cases we have a contradiction, since

$$[f(hb_1,\dots,hb_n),hb_{n+1}]hb_{n+2}=[f(b_1,\dots,b_n),b_{n+1}]b_{n+2}\neq 0$$

and

$$s_4(hc_1,\dots,hc_4)hc_5=s_4(c_1,\dots,c_4)c_5\neq 0. \qquad \blacksquare$$

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